

# Matter Collineations of Some Well Known Spacetimes

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## Abstract

*We derive matter collineations of the Bianchi types I, II, III, VIII and IX, and Kantowski-Sachs spacetimes. It is found that matter collineations turn out similar to Ricci collineations with different constraint equations. We solve the constraint equation for a particular case and obtain three cosmological models which represent perfect fluid dust solutions.*

**PACS:** 04.20Gz, 02.40Ky

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## I. INTRODUCTION

There has been a recent literature [1-5, and references therein] which shows a significant interest in the study of various symmetries. These symmetries arise in the exact solutions of Einstein field equations given by

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab}, \quad (1)$$

where  $G_{ab}$  represents the components of Einstein tensor,  $R_{ab}$  are the components of Ricci tensor and  $T_{ab}$  are the components of matter (or energy-momentum) tensor,  $R$  is the Ricci scalar and  $\kappa$  is the gravitational constant. The geometrical nature of a spacetime is expressed by the metric tensor through Einstein field equations. The well known connection between Killing vectors (KVs) and constants of the motion [6,7] has encouraged the search for general relations between collineations and conservation laws [2].

Curvature and the Ricci tensors are the important quantities which play a vital role in understanding the geometric structure of spacetime. A basic work on curvature collineations (CCs) and Ricci collineations (RCs) has been carried out by Katzin et al. [8] and a complete classification of CCs and RCs for spherical and plane symmetric spacetimes has been obtained by Qadir et al. [2,3].

The energy-momentum tensor represents the matter part of the Einstein field equations and gives the matter field symmetries. Thus the study of matter collineations (MCs) seems more relevant from the physical point of view. Carot et al. [9] have studied MCs, as a symmetry property of the energy-momentum tensor  $T_{ab}$  and have discussed the possible MCs for the case of a degenerate  $T_{ab}$ . Hall et al. [10] have presented a discussion of RCs and MCs in spacetime and have suggested for the evaluation of matter symmetries to see the similarities between RCs and MCs of the spacetime. Recently, Yavuz et al. have discussed the RCs for the Bianchi types II, VIII and IX [11] and for the Bianchi types I and III, and Kantowski-Sachs spacetimes [12]. In this paper, we address the problem of calculating MCs for these spacetimes and establish the relation between KVs, RCs and MCs. We shall

present the complete procedure for solving the MC equations for the Bianchi types I and III, and Kantowski-Sachs spacetimes. As the same methods apply for the Bianchi types II, VIII and IX spacetimes, we do not solve them explicitly and only give the results.

The break of the paper follows. In the next section we shall write down the set of MC equations for the Bianchi types I and III, and Kantowski-Sachs metrics and in section three we shall solve them. In section four, we shall solve MC equations for the Bianchi types II, VIII and IX metrics. Finally, a summary of the results obtained will be presented.

## II. MATTER COLLINEATION EQUATIONS

A vector  $\xi$  is called a MC if the Lie derivative of the energy-momentum tensor along that vector is zero. That is,

$$\mathcal{L}_\xi T = 0, \quad (2)$$

where  $T$  is the energy-momentum tensor and  $\mathcal{L}_\xi$  denotes the Lie derivative along  $\xi$  of the energy-momentum tensor  $T$ . This equation, in a torsion-free space in a coordinate basis, reduces to a simple partial differential equation (PDE),

$$T_{ab,c}\xi^c + T_{ac}\xi_{,b}^c + T_{bc}\xi_{,a}^c = 0, \quad a, b, c = 0, 1, 2, 3. \quad (3)$$

where  $,$  denotes partial derivative with respect to the respective coordinate. These are ten coupled PDEs for four unknown functions ( $\xi^a$ ) which are functions of all spacetime coordinates.

The metric for the Bianchi types I ( $\delta = 0$ ) and III ( $\delta = -1$ ), and Kantowski-Sachs ( $\delta = +1$ ) cosmological models is given by [13]-[15].

$$ds^2 = dt^2 - A^2(t)dr^2 - B^2(t)(d\theta^2 + f^2(\theta)d\phi^2), \quad (4)$$

where  $f(\theta)$  is  $\theta$ ,  $\sinh \theta$  or  $\sin \theta$  according as  $\delta = -\frac{f''}{f} = 0, -1$  or  $+1$  respectively and prime denotes differentiation with respect to  $\theta$ . Notice that  $f^2(\frac{f'}{f})' = -1 \Leftrightarrow (ff')' = 2f'^2 - 1$ .

The non-vanishing components of the energy-momentum tensor for the above metric are the following

$$T_{00} = 2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{\delta}{B^2}, \quad (5)$$

$$T_{11} = -A^2\left(\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{\delta}{B^2}\right), \quad (6)$$

$$T_{22} = -B^2\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB}\right), \quad (7)$$

$$T_{33} = f^2 T_{22}, \quad (8)$$

where dot denotes differentiation with respect to the time coordinate  $t$ . It is to be noted that we have taken  $\kappa = 1$  for simplicity. Using Eqs.(3) and (5)-(8), the MC equations will become

$$(M_{00}) : \quad T_{00,0}\xi^0 + 2T_{00}\xi_{,0}^0 = 0, \quad (9)$$

$$(M_{11}) : \quad T_{11,0}\xi^0 + 2T_{11}\xi_{,1}^1 = 0, \quad (10)$$

$$(M_{22}) : \quad T_{22,0}\xi^0 + 2T_{22}\xi_{,2}^2 = 0, \quad (11)$$

$$(M_{33}) : \quad fT_{22,0}\xi^0 + 2f'T_{22}\xi^2 + 2fT_{22}\xi_{,3}^3 = 0, \quad (12)$$

$$(M_{01}) : \quad T_{00}\xi_{,1}^0 + T_{11}\xi_{,0}^1 = 0, \quad (13)$$

$$(M_{02}) : \quad T_{00}\xi_{,2}^0 + T_{22}\xi_{,0}^2 = 0, \quad (14)$$

$$(M_{03}) : \quad T_{00}\xi_{,3}^0 + f^2T_{22}\xi_{,0}^3 = 0, \quad (15)$$

$$(M_{12}) : \quad T_{11}\xi_{,2}^1 + T_{22}\xi_{,1}^2 = 0, \quad (16)$$

$$(M_{13}) : \quad T_{11}\xi_{,3}^1 + f^2T_{22}\xi_{,1}^3 = 0, \quad (17)$$

$$(M_{23}) : \quad \xi_{,3}^2 + f^2\xi_{,2}^3 = 0. \quad (18)$$

### III. SOLUTION OF THE MC EQUATIONS

We solve the MC equations (9)-(18) for the following three cases:

- (1) One component of  $\xi^a(x^b)$  is different from zero;
- (2) Two components of  $\xi^a(x^b)$  are different from zero;
- (3) Three components of  $\xi^a(x^b)$  are different from zero.

#### A. One Component of $\xi^a(x^b)$ is Different from Zero

This case has the following four possibilities:

- (1a)  $\xi^a = (\xi^0(x^b), 0, 0, 0)$ ;
- (1b)  $\xi^a = (0, \xi^1(x^b), 0, 0)$ ;
- (1c)  $\xi^a = (0, 0, \xi^2(x^b), 0)$ ;
- (1d)  $\xi^a = (0, 0, 0, \xi^3(x^b))$ .

In the case (1a), we have  $\xi^0 = \frac{c_0}{\sqrt{|T_{00}|}}$ , where  $c_0$  is an arbitrary constant. Using Eqs.(10) and (11), it follows that  $T_{11} = \text{constant} = T_{22}$ . Thus, in this case, we have one MC, i.e.,  $\xi_{(1)} = \frac{1}{\sqrt{|T_{00}|}} \frac{\partial}{\partial t}$ . In the cases (1b) and (1d),  $\xi^1$  and  $\xi^3$  respectively become *constants*. For the case (1c), using Eq.(12) we have  $f'T_{22} = 0$  which implies that either  $f$  is constant or  $T_{22} = 0$ . If  $f$  is constant then  $\xi^2$  also becomes *constant* and if  $T_{22} = 0$  then  $\xi^2$  becomes an arbitrary function of  $x^a$ . It follows that in each case (1a-d), MC will be one.

#### B. Two Components of $\xi^a(x^b)$ are Different from Zero

In this case, we have six different possibilities:

- (2a)  $\xi^a = (\xi^0(x^b), \xi^1(x^b), 0, 0)$ ;
- (2b)  $\xi^a = (\xi^0(x^b), 0, \xi^2(x^b), 0)$ ;
- (2c)  $\xi^a = (\xi^0(x^b), 0, 0, \xi^3(x^b))$ ;
- (2d)  $\xi^a = (0, \xi^1(x^b), \xi^2(x^b), 0)$ ;
- (2e)  $\xi^a = (0, \xi^1(x^b), 0, \xi^3(x^b))$ ;

$$(2f) \quad \xi^a = (0, 0, \xi^2(x^b), \xi^3(x^b)).$$

**Case (2a)**  $\xi^a = (\xi^0(x^b), \xi^1(x^b), 0, 0)$ :

In this case, from Eqs.(14)-(17) we see that  $\xi^0$  and  $\xi^1$  become functions of  $t, r$  and from Eq.(9) we have  $\xi^0 = \frac{A(r)}{\sqrt{|T_{00}|}}$ , where  $A(r)$  is an integration function. Using Eq.(11) or (12), we obtain  $T_{22,0}\xi^0 = 0$  which implies that either (i)  $T_{22,0} = 0$  or (ii)  $\xi^0 = 0$ . In the subcase (i), after replacing the value of  $\xi^0$  in Eq.(10), it follows that

$$\xi^1(t, r) = -\frac{T_{11,0}}{2T_{11}\sqrt{|T_{00}|}} \int A(r)dr + B(t), \quad (19)$$

where  $B(t)$  is an integration function. Substituting this value of  $\xi^1$  together with  $\xi^0$  in Eq.(13), we obtain

$$\frac{A_{,11}}{A} = \frac{T_{11}}{\sqrt{|T_{00}|}} \left( \frac{T_{11,0}}{2T_{11}\sqrt{|T_{00}|}} \right) = -\alpha^2, \quad (20)$$

where  $\alpha^2$  is a separation constant which may be positive, negative or zero.

When  $\alpha^2 > 0$ , we have

$$\begin{aligned} \xi^0 &= \frac{1}{\sqrt{|T_{00}|}} (c_0 \cos \alpha r + c_1 \sin \alpha r), \\ \xi^1 &= -\frac{T_{11,0}}{2\alpha T_{11}\sqrt{|T_{00}|}} (c_0 \sin \alpha r - c_1 \cos \alpha r) + c_2, \end{aligned} \quad (21)$$

where  $c_0, c_1, c_2$  are arbitrary constants. It follows that MCs can be written as

$$\begin{aligned} \xi_{(1)} &= \frac{1}{\sqrt{|T_{00}|}} \cos \alpha r \frac{\partial}{\partial t} - \frac{T_{11,0}}{2\alpha T_{11}\sqrt{|T_{00}|}} \sin \alpha r \frac{\partial}{\partial r}, \\ \xi_{(2)} &= \frac{1}{\sqrt{|T_{00}|}} \sin \alpha r \frac{\partial}{\partial t} + \frac{T_{11,0}}{2\alpha T_{11}\sqrt{|T_{00}|}} \cos \alpha r \frac{\partial}{\partial r}, \quad \xi_{(3)} = \frac{\partial}{\partial r}. \end{aligned} \quad (22)$$

When  $\alpha^2 < 0$ ,  $\alpha^2$  is replaced by  $-\alpha^2$  in Eq.(20) and we obtain the following solution

$$\begin{aligned} \xi^0 &= \frac{1}{\sqrt{|T_{00}|}} (c_0 \cosh \alpha r + c_1 \sinh \alpha r), \\ \xi^1 &= -\frac{T_{11,0}}{2\alpha T_{11}\sqrt{|T_{00}|}} (c_0 \sinh \alpha r + c_1 \cosh \alpha r) + c_2. \end{aligned} \quad (23)$$

For  $\alpha^2 = 0$ , we obtain  $\frac{T_{11,0}}{2T_{11}\sqrt{|T_{00}|}} = \beta$ , where  $\beta$  is an arbitrary constant. This implies that either  $\beta$  is non-zero or zero. For  $\beta \neq 0$ , it follows that

$$\xi^0 = \frac{1}{\sqrt{|T_{00}|}}(c_0 r + c_1), \quad \xi^1 = -\beta(c_0 \frac{r^2}{2} + c_1 r) + c_0 \int \frac{\sqrt{|T_{00}|}}{T_{11}} dt + c_2. \quad (24)$$

For  $\beta = 0$ , we have

$$\xi^0 = \frac{1}{\sqrt{|T_{00}|}}(c_0 r + c_1), \quad \xi^1 = \frac{c_0}{T_{11}} \int \sqrt{|T_{00}|} dt + c_2, \quad T_{11} = \text{constant}. \quad (25)$$

Thus, in the subcase (2ai), MCs turn out to be three in all the possibilities. If Eq.(20) is not satisfied by  $T_{00}$  and  $T_{11}$  then  $A = 0$  and this reduces to the case (1b). For the subcase (ii) when  $\xi^0 = 0$ , it also reduces to the case (1b).

**Case (2b)**  $\xi^a = (\xi^0(x^b), 0, \xi^2(x^b), 0)$ :

In this case, we see from Eqs.(13),(15),(16) and (18) that  $\xi^0$  and  $\xi^2$  become functions of  $t, \theta$  and from Eq.(9) we have  $\xi^0 = \frac{A(\theta)}{\sqrt{|T_{00}|}}$ , where  $A(\theta)$  is an integration function. It follows from Eq.(10) that either (i)  $T_{11,0} = 0$  or (ii)  $\xi^0 = 0$ . In the subcase (i), subtracting Eqs.(11) and (12), we have

$$\xi^2(t, \theta) = f B(t), \quad (26)$$

where  $B(t)$  is an integration function. Plugging the values of  $\xi^0$  and  $\xi^2$  in Eq.(11) we obtain

$$\frac{T_{22,0}}{2T_{22}\sqrt{|T_{00}|}} \frac{1}{B} = -\frac{f'}{A} = \alpha^2, \quad (27)$$

where  $\alpha^2$  is a separation constant which is not zero. We can choose  $\alpha^2 = 1$  without loss of generality then the values of  $A$  and  $B$  can be found from Eq.(27). Using these values of  $A$  and  $B$ ,  $\xi^0$  and  $\xi^1$  will take the form

$$\xi^0 = -\frac{f'}{\sqrt{|T_{00}|}}, \quad \xi^2 = \frac{f T_{22,0}}{2T_{22}\sqrt{|T_{00}|}}. \quad (28)$$

If we make use of Eq.(28) in Eq.(14), we obtain

$$\frac{T_{22}}{\sqrt{|T_{00}|}} \left( \frac{T_{22,0}}{2T_{22}\sqrt{|T_{00}|}} \right)' = \frac{f''}{f} = -\delta, \quad (29)$$

where  $\delta$  can take values of 0, -1 or +1 according as Bianchi types I, III or Kantowski-Sachs metrics respectively. For  $\delta = 0$ , we have  $\frac{T_{22,0}}{2T_{22}\sqrt{|T_{00}|}} = \beta$ , where  $\beta$  is an integration constant.

This gives rise to two different possibilities either  $\beta \neq 0$  or  $\beta = 0$ . In the first subcase we have

$$\xi^0 = \frac{c_0}{\sqrt{|T_{00}|}}, \quad \xi^2 = c_1 \beta \theta. \quad (30)$$

Thus the MCs will be two which are  $\xi_{(1)} = \frac{1}{\sqrt{|T_{00}|}} \frac{\partial}{\partial t}$ ,  $\xi_{(2)} = \beta \theta \frac{\partial}{\partial \theta}$ . For the second subcase, we have the results of (1a). The subcase (ii), when  $\xi^0 = 0$ , reduces to the case (1b).

**Case (2c)**  $\xi^a = (\xi^0(x^b), 0, 0, \xi^3(x^b))$ :

In this case, using Eqs.(13),(14),(17) and (18), it turns out that  $\xi^0$  and  $\xi^3$  are functions of  $t, \phi$  and from Eq.(9), it follows that  $\xi^0 = \frac{A(\phi)}{\sqrt{|T_{00}|}}$ , where  $A(\phi)$  is an integration function. From Eq.(10) or (11) we see that either  $T_{11,0} = 0 = T_{22,0}$  or  $\xi^0 = 0$ . In the first subcase, we have

$$\xi^0 = \frac{c_0}{\sqrt{|T_{00}|}}, \quad \xi^3 = c_1. \quad (31)$$

It follows that the MCs turn out to be two, i.e.,  $\xi_{(1)} = \frac{1}{\sqrt{|T_{00}|}} \frac{\partial}{\partial t}$ ,  $\xi_{(2)} = \frac{\partial}{\partial \phi}$ . The second subcase reduces to (1c).

**Case (2d)**  $\xi^a = (0, \xi^1(x^b), \xi^2(x^b), 0)$ :

In this case, we see from Eqs.(10), (11), (13), (14), (17) and (18) that  $\xi^1$  and  $\xi^2$  become functions of  $\theta$  and  $r$  respectively. Eq.(12) implies that either  $T_{22} = 0$  or  $\xi^2 = 0$ . If  $T_{22} = 0$ , we have  $\xi^1 = c_0$ ,  $\xi^2 = \xi(x^a)$ . Thus MCs are  $\xi = \frac{\partial}{\partial r} + \xi^2(x^a) \frac{\partial}{\partial \theta}$  which shows that MCs are arbitrary in the  $\theta$  direction. When  $T_{11} = 0$  then both  $\xi^1$  and  $\xi^2$  become arbitrary functions of four vector  $x^a$ . It follows that  $\xi = \xi^1(x^a) \frac{\partial}{\partial r} + \xi^2(x^a) \frac{\partial}{\partial \theta}$  which gives arbitrary MCs in both radial and  $\theta$  directions. In the second subcase, when  $\xi^2 = 0$ , it reduces to (1b) if  $T_{11} \neq 0$  and for  $T_{11} = 0$ ,  $\xi^1$  become arbitrary function of  $x^a$  which gives  $\xi = \xi^1(x^a) \frac{\partial}{\partial r}$ .

When  $T_{11} = 0 = T_{22}$ , using Eqs.(6) and (7), it follows that

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{\delta}{B^2} = 0. \quad (32)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = 0. \quad (33)$$



Plugging the value of  $\frac{\ddot{B}}{B}$  from Eq.(33) in Eq.(32), we obtain

$$2\frac{\ddot{A}}{A} + 2\frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}^2}{B^2} - \frac{\delta}{B^2} = 0. \quad (34)$$

This is a second order non-linear differential equation in  $A$  and  $B$  and can only be solved by assuming some relation between these two functions. If we choose  $B = cA$ , where  $c$  is an arbitrary constant, then Eq.(34) will become

$$2A\ddot{A} + \dot{A}^2 = \frac{\delta}{c^2}. \quad (35)$$

Now assume that  $A = (at+b)^m$ , where  $a, b$  and  $m$  are arbitrary constants, then using Eq.(35), it follows that only possible solutions for  $\delta = 0, +1$  and  $-1$  are  $m = \frac{2}{3}, m = 1(a^2c^2 = 1)$  and  $m = 1(a^2c^2 = -1)$  respectively. Thus for the Bianchi type I ( $\delta = 0$ ) metric, we have

$$ds^2 = dt^2 - (at+b)^{\frac{2}{3}}dr^2 - c(at+b)^{\frac{2}{3}}(d\theta^2 + \theta^2d\phi^2). \quad (36)$$

For the Bianchi type III ( $\delta = -1$ ) metric, it follows that

$$ds^2 = dt^2 - (at+b)dr^2 - c(at+b)(d\theta^2 + \sinh^2\theta d\phi^2), \quad (37)$$

where  $a^2c^2 = -1$ . For the Kantowski-Sachs ( $\delta = +1$ ) metric, we obtain

$$ds^2 = dt^2 - (at+b)dr^2 - c(at+b)(d\theta^2 + \sin^2\theta d\phi^2), \quad (38)$$

where  $a^2c^2 = +1$ . It can easily be verified that all these spacetimes represent perfect fluid dust solutions. The energy density for each of the above metrics are given as

$$\rho_0 = \frac{4a^2}{3(at+b)^2}, \quad \rho_{-1} = \frac{4+3(at+b)^{\frac{2}{3}}}{3(at+b)^2}, \quad \rho_{+1} = -\frac{4+3(at+b)^{\frac{2}{3}}}{3(at+b)^2}. \quad (39)$$

It is interesting to note that the energy density is positive for the Bianchi types I and III metrics.

**Case (2e)**  $\xi^a = (0, \xi^1(x^b), 0, \xi^3(x^b))$ :

It follows from Eqs.(10),(12),(13),(15),(16) and (18) that  $\xi^1$  and  $\xi^3$  are functions of  $\phi$  and  $r$  respectively. Using Eq.(17), it turns out that  $\xi^1$  and  $\xi^3$  become *constants*. Thus the

two MCs will be  $\xi_{(1)} = \frac{\partial}{\partial r}$ ,  $\xi_{(2)} = \frac{\partial}{\partial \phi}$ .

**Case (2f)**  $\xi^a = (0, 0, \xi^2(x^b), \xi^3(x^b))$ :

In this case, using Eqs.(11), (12), (14)-(17) we find that there arise two possible situations either  $T_{22} = 0$  or  $T_{22} \neq 0$ . For the first option,  $\xi^2$  and  $\xi^3$  become arbitrary functions of  $x^a$ . For the second option, it follows from Eqs.(11), (14)-(17) that  $\xi^2 = \xi^2(\phi)$  and  $\xi^3 = \xi^3(\theta, \phi)$ . Using Eqs.(12) and (18), it can be shown, after some algebra, that

$$\xi^2 = c_0 \cos \phi + c_1 \sin \phi, \quad \xi^3 = -\frac{f'}{f}(c_0 \sin \phi - c_1 \cos \phi) + c_2. \quad (40)$$

Thus the MCs are given as

$$\xi = c_0(\cos \phi \frac{\partial}{\partial \theta} - \frac{f'}{f} \sin \phi \frac{\partial}{\partial \phi}) + c_1(\sin \phi \frac{\partial}{\partial \theta} + \frac{f'}{f} \cos \phi \frac{\partial}{\partial \phi}) + c_2 \frac{\partial}{\partial \phi}. \quad (41)$$

It follows that the MCs are three given by

$$\xi_{(1)} = \cos \phi \frac{\partial}{\partial \theta} - \frac{f'}{f} \sin \phi \frac{\partial}{\partial \phi}, \quad \xi_{(2)} = \sin \phi \frac{\partial}{\partial \theta} + \frac{f'}{f} \cos \phi \frac{\partial}{\partial \phi}, \quad \xi_{(3)} = \frac{\partial}{\partial \phi}. \quad (42)$$

which also represent the generators of a group  $G_3$ . These are just the KVs associated with spherical symmetry of the Bianchi types I and III, and Kantowski-Sachs spacetimes for  $\delta = 0, -1, +1$  respectively [13].

### C. Three Components of $\xi^a(x^b)$ are Different from Zero

It has four different possibilities:

$$(3a) \quad \xi^a = (\xi^0(x^b), \xi^1(x^b), \xi^2(x^b), 0);$$

$$(3b) \quad \xi^a = (\xi^0(x^b), \xi^1(x^b), 0, \xi^3(x^b));$$

$$(3c) \quad \xi^a = (\xi^0(x^b), 0, \xi^2(x^b), \xi^3(x^b));$$

$$(3d) \quad \xi^a = (0, \xi^1(x^b), \xi^2(x^b), \xi^3(x^b));$$

**Case (3a)**  $\xi^a = (\xi^0(x^b), \xi^1(x^b), \xi^2(x^b), 0)$ :

In this case, using Eqs.(15), (17) and (18) we find that  $\xi^0, \xi^1$  and  $\xi^2$  are functions of  $t, r, \theta$  and from Eq.(9) we have  $\xi^0 = \frac{A(r, \theta)}{\sqrt{|T_{00}|}}$ , where  $A(r, \theta)$  is an integration function. If we make

use of this value of  $\xi^0$  in Eq.(12), we obtain  $\xi^2 = -\frac{T_{22,0}}{2T_{22}\sqrt{|T_{00}|}}\frac{Af'}{f'}$ . Substituting this value of  $\xi^2$  together with  $\xi^0$  in Eq.(11), we obtain the value of  $A$  as follows

$$A(r, \theta) = A_1(r)f', \quad (43)$$

where  $A_1(r)$  is an integration function. From Eq.(10), it follows that

$$\xi^1 = -\frac{T_{11,0}}{2T_{11}\sqrt{|T_{00}|}}f' \int A_1 dr + B(t, \theta), \quad (44)$$

where  $B(t, \theta)$  is an integration function. Using values of  $\xi^0$  and  $\xi^2$  in Eq.(14), we have

$$\frac{T_{22}}{\sqrt{|T_{00}|}}\left(\frac{T_{22,0}}{2T_{22}\sqrt{|T_{00}|}}\right) = \frac{f''}{f} = -\delta. \quad (45)$$

Now plugging the values of  $\xi^0$  and  $\xi^1$  in Eq.(13), we obtain

$$\frac{T_{11}}{\sqrt{|T_{00}|}}\left(\frac{T_{11,0}}{2T_{11}\sqrt{|T_{00}|}}\right) = \frac{A_{1,11}}{A_1} = -\alpha^2, \quad (46)$$

where  $\alpha^2$  is a separation constant which may be positive, negative or zero. Thus there arise six different possibilities. (i)  $\alpha^2 > 0, \delta \neq 0$ ; (ii)  $\alpha^2 < 0, \delta \neq 0$ ; (iii)  $\alpha^2 = 0, \delta \neq 0$ ; (iv)  $\alpha^2 > 0, \delta = 0$ ; (v)  $\alpha^2 < 0, \delta = 0$ ; (vi)  $\alpha^2 = 0, \delta = 0$ .

For the subcase (i), after some algebraic manipulation, it is shown that

$$\begin{aligned} \xi^0 &= \frac{1}{\sqrt{|T_{00}|}}(c_0 \cos \alpha r + c_1 \sin \alpha r)f', \\ \xi^1 &= -\frac{f'}{\alpha} \frac{T_{11,0}}{2T_{11}\sqrt{|T_{00}|}}(c_0 \sin \alpha r - c_1 \cos \alpha r) + c_2, \\ \xi^2 &= -\frac{T_{22,0}}{2T_{22}\sqrt{|T_{00}|}}(c_0 \cos \alpha r + c_1 \sin \alpha r)f, \quad T_{22} = \frac{\delta}{\alpha^2}T_{11} + c, \end{aligned} \quad (47)$$

where  $c$  is an arbitrary constants and  $\delta$  can take values  $\pm 1$ . It follows that MCs are three which can be written as

$$\begin{aligned} \xi_{(1)} &= \frac{f'}{\sqrt{|T_{00}|}} \cos \alpha r \frac{\partial}{\partial t} - \frac{f'}{\alpha} \frac{T_{11,0}}{2T_{11}\sqrt{|T_{00}|}} \sin \alpha r \frac{\partial}{\partial r} - \frac{fT_{22,0}}{2T_{22}\sqrt{|T_{00}|}} \cos \alpha r \frac{\partial}{\partial \theta}, \\ \xi_{(2)} &= \frac{f'}{\sqrt{|T_{00}|}} \sin \alpha r \frac{\partial}{\partial t} + \frac{f'}{\alpha} \frac{T_{11,0}}{2T_{11}\sqrt{|T_{00}|}} \cos \alpha r \frac{\partial}{\partial r} - \frac{fT_{22,0}}{2T_{22}\sqrt{|T_{00}|}} \sin \alpha r \frac{\partial}{\partial \theta}, \quad \xi_{(3)} = \frac{\partial}{\partial r}. \end{aligned} \quad (48)$$

In the subcase (ii),  $\alpha^2$  is replaced by  $-\alpha^2$  in Eq.(38) and it follows that

$$\begin{aligned}\xi^0 &= \frac{1}{\sqrt{|T_{00}|}}(c_0 \cosh \alpha r + c_1 \sinh \alpha r)f', \\ \xi^1 &= -\frac{f'}{\alpha} \frac{T_{11,0}}{2T_{11}\sqrt{|T_{00}|}}(c_0 \sinh \alpha r + c_1 \cosh \alpha r) + c_2, \\ \xi^2 &= -\frac{T_{22,0}}{2T_{22}\sqrt{|T_{00}|}}(c_0 \cosh \alpha r + c_1 \sinh \alpha r)f, \quad T_{22} = \frac{\delta}{\alpha^2}T_{11} + c.\end{aligned}\tag{49}$$

For the third subcase, we have  $A_1 = c_0 r + c_1$  and  $\frac{T_{11,0}}{2T_{11}\sqrt{|T_{00}|}} = \beta$ , where  $\beta$  is an arbitrary constant. This implies that either  $\beta$  is non-zero or zero. If  $\beta \neq 0$ , it reduces to the case (1b). For  $\beta = 0$ , we have  $T_{11} = \text{constant}$ . Using these values in Eq.(13) and (16) we obtain

$$\begin{aligned}\xi^0 &= \frac{1}{\sqrt{|T_{00}|}}(c_0 r + c_1)f', \quad \xi^1 = -\frac{c_0 f' T_{22,0}}{2\delta T_{11}\sqrt{|T_{00}|}} + c_2, \quad \xi^2 = -\frac{T_{22,0}f}{2T_{22}\sqrt{|T_{00}|}}(c_0 r + c_1), \\ T_{11} &= \text{constant}, \quad T_{22,0} = 2\delta\sqrt{|T_{00}|} \int \sqrt{|T_{00}|} dt.\end{aligned}\tag{50}$$

The fourth and fifth subcases reduce to (1b).

In the subcase (vi), we obtain  $A_1 = c_0 r + c_1$ ,  $\frac{T_{11,0}}{2T_{11}\sqrt{|T_{00}|}} = \beta$  and  $\frac{T_{22,0}}{2T_{22}\sqrt{|T_{00}|}} = \gamma$ , where  $\beta$  and  $\gamma$  are arbitrary constants. We see from Eq.(16) that  $\gamma T_{22} = 0$  which implies that either  $\gamma = 0$  or  $T_{22} = 0$ . This gives rise to three possibilities that (★)  $\gamma \neq 0 \neq \beta, T_{22} = 0$ , (†)  $\gamma = 0, \beta \neq 0 \neq T_{22}$  and (‡)  $\gamma = 0 = \beta, T_{22} \neq 0$ . In the subcase (vi★), for the Bianchi type I spacetime, we have

$$\xi^0 = \frac{1}{\sqrt{|T_{00}|}}(c_0 r + c_1), \quad \xi^1 = -\beta(c_0 \frac{r^2}{2} + c_1 r) - c_0 \int \frac{\sqrt{|T_{00}|}}{T_{11}} dt + c_2, \quad \xi^2 = -\gamma \theta(c_0 r + c_1).\tag{51}$$

For the Bianchi type II and Kantowski-Sachs spacetimes  $c_0 = 0$ . The subcases (vi†) and (vi‡) reduce to the case (2a). In the former subcase  $T_{11}$  becomes *constant* while in the latter subcase both  $T_{11}$  and  $T_{22}$  become *constants*.

**Case (3b)**  $\xi^a = (\xi^0(x^b), \xi^1(x^b), 0, \xi^3(x^b))$ :

In this case, using Eqs.(14), (16) and (18), it follows that  $\xi^0, \xi^1$  and  $\xi^3$  are functions of  $t, r, \phi$ . Using Eqs.(15) and (17), it can be seen that  $\xi^0$  and  $\xi^1$  become functions of  $t, r$  while

$\xi^3$  becomes only a function of  $\phi$ . Further from Eq.(12), it follows that  $\xi^3$  becomes *constant* and  $T_{22,0}\xi^0 = 0$  which implies that either  $T_{22,0} = 0$  or  $\xi^0 = 0$ . If  $\xi^0 = 0$ , this reduces to the case (2e). However, for  $\xi^0 \neq 0$  ( $T_{22,0} = 0$ ), it follows that it gives the results of (2a) together with  $\xi^3 = \text{constant}$ .

**Case (3c)**  $\xi^a = (\xi^0(x^b), 0, \xi^2(x^b), \xi^3(x^b))$ :

In this case, we see from Eqs.(13), (16) and (17) that  $\xi^0, \xi^2$  and  $\xi^3$  are functions of  $t, \theta, \phi$ . Further from Eq.(10) we have either  $T_{11,0} = 0$  or  $\xi^0 = 0$ . After some algebra, it can be shown that both cases reduce to the case (2f).

**Case (3d)**  $\xi^a = (0, \xi^1(x^b), \xi^2(x^b), \xi^3(x^b))$ :

In this case, using Eqs.(10), (11), (13)-(15) we find that  $\xi^1, \xi^2$  and  $\xi^3$  are functions of  $(\theta, \phi), (r, \phi)$  and  $r, \theta, \phi$  respectively. Eq.(16) gives

$$\xi^1 = A_1(\phi)\theta + A_2(\phi), \quad \xi^2 = -arA_1(\phi) + A_3(\phi), \quad T_{11} = aT_{22}, \quad (52)$$

where  $A_1(\phi)$  is an integration function and  $a$  is an arbitrary constant. Using Eq.(17), it follows that

$$\xi^3 = -\frac{ar}{f}(A_{,3}\theta + A_{2,3}) + B_1(\theta, \phi), \quad (53)$$

where  $B_1(\theta, \phi)$  is an integration function. If  $a = 0$ , then  $T_{11} = 0$  and it follows that  $\xi^1$  is an arbitrary function of four vector  $x^a$  and  $\xi^2, \xi^3$  will remain the same as in the case (2f).

If  $a \neq 0$  and  $\delta = 0$ , for the Bianchi type I metric, we have from Eq.(12)

$$A_1 = c_0 \cos \phi + c_1 \sin \phi, \quad A_2 = c_2 \phi + c_3, \quad B_1 = -\frac{1}{\theta} \int A_3 d\phi + A_4(\theta), \quad (54)$$

where  $A_4(\theta)$  is an integration function. Using Eq.(18), we have

$$\begin{aligned} \xi^1 &= (c_0 \cos \phi + c_1 \sin \phi)\theta + c_2, \\ \xi^2 &= -ar(c_0 \cos \phi + c_1 \sin \phi) + c_3 \cos \phi + c_4 \sin \phi, \\ \xi^3 &= \frac{ar}{\theta}(c_0 \sin \phi - c_1 \cos \phi) - \frac{1}{\theta}(c_3 \sin \phi - c_4 \cos \phi) + c_5, \end{aligned} \quad (55)$$

where  $c_i (i = 0, 1, 2, 3, 4, 5)$  are arbitrary constants. Thus the Lie algebra of MCs are spanned by

$$\begin{aligned}\xi_{(1)} &= \cos \phi \frac{\partial}{\partial \theta} - \frac{1}{\theta} \sin \phi \frac{\partial}{\partial \phi}, & \xi_{(2)} &= \sin \phi \frac{\partial}{\partial \theta} + \frac{1}{\theta} \cos \phi \frac{\partial}{\partial \phi}, & \xi_{(3)} &= \frac{\partial}{\partial \phi}, \\ \xi_{(4)} &= \theta \cos \phi \frac{\partial}{\partial r} - ar \xi_{(1)}, & \xi_{(5)} &= \theta \sin \phi \frac{\partial}{\partial r} - ar \xi_{(2)}, & \xi_{(6)} &= \frac{\partial}{\partial r}.\end{aligned}\quad (56)$$

Notice that MCs  $\xi_{(1)}, \xi_{(2)}$  and  $\xi_{(3)}$  correspond to KVs associated with spherical symmetry of the Bianchi type I spacetime while  $\xi_{(4)}, \xi_{(5)}$  and  $\xi_{(6)}$  are the proper MCs of the Bianchi type I spacetime. The non-vanishing commutators are given as

$$\begin{aligned}[\xi_{(1)}, \xi_{(3)}] &= \xi_{(2)}, & [\xi_{(2)}, \xi_{(3)}] &= -\xi_{(1)}, & [\xi_{(1)}, \xi_{(4)}] &= [\xi_{(2)}, \xi_{(5)}] = \xi_{(6)}, \\ [\xi_{(1)}, \xi_{(6)}] &= [\xi_{(3)}, \xi_{(5)}] = \xi_{(4)}, & [\xi_{(3)}, \xi_{(4)}] &= -\xi_{(5)}, & [\xi_{(4)}, \xi_{(6)}] &= a\xi_{(1)}, \\ [\xi_{(4)}, \xi_{(5)}] &= -a\xi_{(3)}, & [\xi_{(5)}, \xi_{(6)}] &= a\xi_{(2)}.\end{aligned}\quad (57)$$

If  $a \neq 0 \neq \delta$ , for the Bianchi type III and Kantowski-Sachs spacetimes, then from Eq.(12) we have  $A_1 = 0, A_2 = c_0\phi + c_1$  and  $B_1 = -\frac{f'}{f} \int A_3 d\phi + A_4(\theta)$ . Using Eq.(18), it follows that

$$\xi^1 = c_0, \quad \xi^2 = c_1 \cos \phi + c_2 \sin \phi, \quad \xi^3 = -\frac{f'}{f} (c_1 \sin \phi - c_2 \cos \phi) + c_3. \quad (58)$$

The MCs will, therefore, become

$$\xi_{(1)} = \cos \phi \frac{\partial}{\partial \theta} - \frac{f'}{f} \sin \phi \frac{\partial}{\partial \phi}, \quad \xi_{(2)} = \sin \phi \frac{\partial}{\partial \theta} + \frac{f'}{f} \cos \phi \frac{\partial}{\partial \phi}, \quad \xi_{(3)} = \frac{\partial}{\partial \phi}, \quad \xi_{(4)} = \frac{\partial}{\partial r}. \quad (59)$$

which are also the generators of a group  $G_4$ . We see that  $\xi_{(1)}, \xi_{(2)}$  and  $\xi_{(3)}$  are the same as in (2f). Thus the MCs  $\xi_{(1)}, \xi_{(2)}, \xi_{(3)}$  and  $\xi_{(4)}$  are non-proper.

#### IV. MCS OF THE BIANCHI TYPES II, VIII, IX SPACETIMES

The locally rotationally symmetric metric for the spatially homogeneous Bianchi types II ( $\delta = 0$ ), VIII ( $\delta = -1$ ) and IX ( $\delta = +1$ ) cosmological models can generally be written in the form [16,17]

$$ds^2 = dt^2 - S^2(t)(dx - h(y)dz)^2 - R^2(t)(dy^2 + f^2(y)dz^2), \quad (60)$$

where  $f(y)$  is  $y, \sin y$  or  $\sinh y$  according as  $\delta = -\frac{f''}{f} = 0, +1$  or  $-1$  respectively and  $h(y)$  is  $-\frac{y}{2}, \cos y$  or  $-\cosh y$  for  $\delta = 0, +1$  or  $-1$  respectively. Here prime denotes differentiation with respect to  $y$ . The non-vanishing components of energy-momentum tensor are given as follows

$$T_{00} = 2\frac{\dot{R}\dot{S}}{RS} + \frac{\dot{R}^2}{R^2} - \frac{S^2}{4R^4} - \frac{\delta}{R^2}, \quad (61)$$

$$T_{11} = -2\frac{\ddot{R}S^2}{R} - \frac{\dot{R}^2S^2}{R^2} + 3\frac{S^4}{4R^4} - \delta\frac{S^2}{R^2}, \quad (62)$$

$$T_{22} = -R\ddot{R} - \frac{R^2\ddot{S}}{S} - \frac{R\dot{R}\dot{S}}{S} - \frac{S^2}{4R^2}, \quad (63)$$

$$T_{33} = h^2T_{11} + f^2T_{22}, \quad (64)$$

$$T_{13} = -hT_{11}. \quad (65)$$

The MC equations will become

$$(M_{00}) : \quad T_{00,0}\xi^0 + 2T_{00}\xi_{,0}^0 = 0, \quad (66)$$

$$(M_{11}) : \quad T_{11,0}\xi^0 + 2(T_{11}\xi_{,1}^1 - hT_{11}\xi_{,1}^3) = 0, \quad (67)$$

$$(M_{22}) : \quad T_{22,0}\xi^0 + 2T_{22}\xi_{,2}^2 = 0, \quad (68)$$

$$(M_{33}) : \quad T_{33,0}\xi^0 + T_{33,2}\xi^2 + 2(-hT_{11}\xi_{,3}^1 + T_{33}\xi_{,3}^3) = 0, \quad (69)$$

$$(M_{01}) : \quad T_{00}\xi_{,1}^0 + T_{11}\xi_{,0}^1 - hT_{11}\xi_{,0}^3 = 0, \quad (70)$$

$$(M_{02}) : \quad T_{00}\xi_{,2}^0 + T_{22}\xi_{,0}^2 = 0, \quad (71)$$

$$(M_{03}) : \quad T_{00}\xi_{,3}^0 - hT_{11}\xi_{,0}^1 + T_{33}\xi_{,0}^3 = 0, \quad (72)$$

$$(M_{12}) : \quad T_{11}\xi_{,2}^1 - hT_{11}\xi_{,2}^3 + T_{22}\xi_{,1}^2 = 0, \quad (73)$$

$$(M_{13}) : \quad hT_{11,0}\xi^0 + h'T_{11}\xi^2 - T_{11}\xi_{,3}^1 + hT_{11}\xi_{,3}^3 + hT_{11}\xi_{,1}^1 - T_{33}\xi_{,1}^3 = 0, \quad (74)$$

$$(M_{23}) : \quad T_{22}\xi_{,3}^2 - hT_{11}\xi_{,2}^1 + T_{33}\xi_{,2}^3 = 0. \quad (75)$$

We solve the above equations for the cases (1), (2), (3b) and (3d) given in the last section.

In all the possibilities of the case (1), we have the similar results as in the previous section for the Bianchi types I, III and Kantowski-Sachs spacetimes.

In the case (2a), using Eq.(68), it follows that either  $T_{22,0} = 0$  or  $\xi^0 = 0$ . For  $T_{22,0} = 0$ , after some algebra, we obtain  $\xi^0 = \frac{c_0}{\sqrt{|T_{00}|}}$ ,  $\xi^1 = c_1$  and  $T_{11} = \text{constant}$ . The subcase  $\xi^0 = 0$  reduces to the case (1b). In the case (2b), using Eq.(67), we see that either  $T_{11,0} = 0$  or  $\xi^0 = 0$ . When we take  $T_{11,0} = 0$ , it gives  $\xi^2 = 0$  and so reduces to the case (1a) while  $\xi^0 = 0$  reduces to (1c). In the case (2c), Eq.(68) yields that either  $T_{22,0} = 0$  or  $\xi^0 = 0$ . If  $T_{22,0} = 0$ , after some algebraic manipulation, it follows that  $\xi^0 = \frac{c_0}{\sqrt{|T_{00}|}}$ ,  $\xi^3 = c_1$  and  $T_{11} = \text{constant}$  while the subcase  $\xi^0 = 0$  reduces to the case (1d). In the case (2d), we employ Eqs.(69) and (74) which show that either  $T_{11} = 0 = T_{22}$  or  $\xi^2 = 0$ . For the former subcase, we obtain  $\xi^1$  and  $\xi^2$  as an arbitrary functions of four vector  $x^a$  while for the latter subcase, we have the result of (1b). In the case (2e), after some algebra, we obtain  $\xi^1 = c_0$ ,  $\xi^3 = c_1$ . The case (2f) gives the similar results as the case (1d).

In the case (3b), it follows from Eq.(68) that either  $T_{22,0} = 0$  or  $\xi^0 = 0$ . For  $T_{22,0} = 0$ , we obtain  $\xi^0 = \frac{c_0}{\sqrt{|T_{00}|}}$ ,  $\xi^1 = c_1$ ,  $\xi^3 = c_2$  and  $T_{11} = \text{constant}$  while the subcase  $\xi^0 = 0$  reduces to the case (2e). In the case (3d), if we choose  $T_{11} = aT_{22}$ , where  $a$  is an arbitrary constant then we have two possibilities either  $a = 0$  or  $a \neq 0$ . For  $a = 0$ , after some algebraic manipulation, we obtain

$$\xi^1 = \xi^1(y, z), \quad \xi^2 = c_0 \cos z + c_1 \sin z, \quad \xi^3 = -\frac{f'}{f}(c_0 \sin z - c_1 \cos z) + c_2 \quad (76)$$

and for  $a \neq 0$ , this reduces to the case (2e).



## V. CONCLUSION

We have evaluated the MCs for the Bianchi types I, II, III, VIII and IX, and Kantowski-Sachs spacetimes. First we discuss the MCs for the Bianchi types I and III, and Kantowski-Sachs spacetimes. We see that in the cases (1b-d), the MCs are identical to the KVs. Further we note that in the case (1b), the MC represents a translation along the radial direction  $r$  while in the cases (1c) and (1d), the MC represents rotations in  $\theta$  and  $\phi$  directions respectively.

We know that every KV is an MC, but the converse is not always true. As given by Carot et al. [4], if  $T_{ab}$  is non-degenerate,  $\det(T_{ab}) \neq 0$ , the Lie algebra of the MCs is finite dimensional. If  $T_{ab}$  is degenerate, i.e.,  $\det(T_{ab}) = 0$ , we cannot guarantee the finite dimensionality of the MCs. Thus for the Bianchi types I and III, and Kantowski-Sachs cosmological models, the Lie algebra of MCs is finite dimensional if  $T_{ab}$  is non-degenerate and may be infinite dimensional if  $T_{ab}$  is degenerate. Now  $T_{ab}$  is degenerate if any of  $T_{00}, T_{11}$  and  $T_{22}$  vanishes. Since in case (2d)  $T_{22}\xi^0 = 0$ , that Lie algebra of MCs is infinite dimensional. Further, in this case, it should be noted that we have obtained MCs identical to KVs for the non-degenerate  $T_{ab}$ . We observe that the cases (1a), (2a-c), (3a-c), which contain the component  $\xi^0$ , have proper MCs and the cases (2e) and (2f), which do not contain  $\xi^0$ , have non-proper MCs. Also, it follows from the case (3d) that for the Bianchi type I spacetime, some of the MCs are proper but for the Bianchi type III and Kantowski-Sachs metrics, MCs are non-proper.

For the Bianchi types II, VIII and IX, spacetimes, we have one MC in the case (1). Here  $\xi^i (i = 1, 2, 3)$  gives a translation along  $x, y$  and  $z$  directions respectively. The cases (2a) and (2c) either give two or one MC according to the constraint while the cases (2b) and (2f) yield one. In the case (2d), we have either arbitrary MCs or one according to the constraint whereas in the case (2e), we get two. Similarly, the case (3b) either give three or two MCs and the case (3d) either yield  $3 + \xi^1(y, z)$  or two. For these cosmological models, it should be noted that the cases (1a), (2a-c), (3b), types of symmetry vectors are proper MCs while

the cases (1b), (1c), (2d-f) and (3d) give to non-proper MCs.

We remark that RCs obtained by Yavuz et al. [11,12] are similar to MCs. However, the constraint equations are different. If we solve these constraint equations, we may have a family of spacetimes. We have obtained three cosmological models which turn out to perfect fluid dust solutions. It would be interesting to look for further solutions from these constraint equations.

## Acknowledgments

The author would like to thank Prof. Chul H. Lee for his hospitality at the Department of Physics and Korea Scientific and Engineering Foundation (KOSEF) for postdoc fellowship at Hanyang University Seoul, KOREA. I am also grateful to Prof. Asghar Qadir for his useful comments during its write up.

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